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# On dual processes of non-symmetric diffusions with measure-valued drifts

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## Abstract

For $\mu = (\mu^1, \ldots, \mu^d)$ with each $\mu^i$ being a signed measure on $\mathbf{R}^d$ belonging to the Kato class $\mathbf{K}_{d,1}$, a diffusion with drift $\mu$ is a diffusion process in $\mathbf{R}^d$ whose generator can be formally written as $L + \mu \cdot \nabla$ where $L$ is a uniformly elliptic differential operator. When each $\mu^i$ is given by $U^i(x)\mathrm{d}x$ for some function $U^i$, a diffusion with drift $\mu$ is a diffusion in $\mathbf{R}^d$ with generator $L + U \cdot \nabla$. In [P. Kim, R. Song, Two-sided estimates on the density of Brownian motion with singular drift, Illinois J. Math. 50 (2006) 635–688; P. Kim, R. Song, Boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains, Math. Ann., 339 (1) (2007) 135–174], we have already studied properties of diffusions with measure-valued drifts in bounded domains. In this paper we first show that the killed diffusion process with measure-valued drift in any bounded domain has a dual process with respect to a certain reference measure. We then discuss the potential theory of the dual process and Schrödinger-type operators of a diffusion with measure-valued drift. More precisely, we prove that (1) for any bounded domain, a scale invariant Harnack inequality is true for the dual process; (2) if the domain is bounded $C^{1,1}$, the boundary Harnack principle for the dual process is valid and the (minimal) Martin boundary for the dual process can be identified with the Euclidean boundary; and (3) the harmonic measure for the dual process is locally comparable to that of the $h$-conditioned Brownian motion with $h$ being an eigenfunction corresponding to the largest Dirichlet eigenvalue in the domain.

The Schrödinger operator that we consider can be formally written as $L+\mu\cdot\nabla+\nu$ where $L$ is uniformly elliptic, $\mu$ is a vector-valued signed measure on $\mathbf{R}^d$ and $\nu$ is a signed measure in $\mathbf{R}^d$. We show that, for a bounded Lipschitz domain and under the gaugeability assumption, the (minimal) Martin boundary for

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the Schrödinger operator obtained from the diffusion with measure-valued drift can be identified with the Euclidean boundary.
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## 1. Introduction

In this paper, we continue our discussion of diffusions with measure-valued drift from [16, 17].

Throughout this paper, we always assume that $d \geq 3$. First we recall the definition of the Kato class $\mathbf{K}_{d,\alpha}$ for $\alpha \in (0, 2]$. For any function $f$ on $\mathbf{R}^d$ and $r > 0$, we define

$$M_f^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \frac{|f(y)|\mathrm{d}y}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leq 2.$$

In this paper, we mean, by a signed measure, the difference of two non-negative measures at most one of which can have infinite total mass. For any signed measure $\nu$ on $\mathbf{R}^d$, we use $\nu^+$ and $\nu^-$ to denote its positive and negative parts, and $|\nu| = \nu^+ + \nu^-$ its total variation. For any signed measure $\nu$ on $\mathbf{R}^d$ and any $r > 0$, we define

$$M_\nu^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \frac{|\nu|(\mathrm{d}y)}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leq 2.$$

**Definition 1.1.** Let $0 < \alpha \leq 2$. We say that a function $f$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_f^\alpha(r) = 0$. We say that a signed Radon measure $\nu$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_\nu^\alpha(r) = 0$. We say that a $d$-dimensional vector-valued function $V = (V^1, \dots, V^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each $V^i$ belongs to $\mathbf{K}_{d,\alpha}$. We say that a $d$-dimensional vector-valued signed Radon measure $\mu = (\mu^1, \dots, \mu^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each $\mu^i$ belongs to $\mathbf{K}_{d,\alpha}$.

Rigorously speaking a function $f$ in $\mathbf{K}_{d,\alpha}$ may not give rise to a signed measure $\nu$ in $\mathbf{K}_{d,\alpha}$ since it may not give rise to a signed measure at all. However, for the sake of simplicity we use the convention that whenever we write that a signed measure $\nu$ belongs to $\mathbf{K}_{d,\alpha}$ we are implicitly assuming that we are covering the case of all the functions in $\mathbf{K}_{d,\alpha}$ as well.

Throughout this paper we assume that $\mu = (\mu^1, \dots, \mu^d)$ is fixed with each $\mu^i$ being a signed measure on $\mathbf{R}^d$ belonging to $\mathbf{K}_{d,1}$. We also assume that the operator $L$ is either $L_1$ or $L_2$ where

$$L_1 := \frac{1}{2} \sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) \quad \text{and} \quad L_2 := \frac{1}{2} \sum_{i,j=1}^d a_{ij}\partial_i\partial_j$$

with $\mathbf{A} := (a_{ij})$ being $C^1$ and uniformly elliptic but not necessarily symmetric.

Formally speaking, when $a_{ij}$ is symmetric, a diffusion process in $\mathbf{R}^d$ with drift $\mu$ is a diffusion process in $\mathbf{R}^d$ with generator $L + \mu \cdot \nabla$. When each $\mu^i$ is given by $U^i(x)\mathrm{d}x$ for some function

$U^i$, a diffusion process with drift $\mu$ is a diffusion in $\mathbf{R}^d$ with generator $L + U \cdot \nabla$ and it is a solution to the SDE $\mathrm{d}X_t = \mathrm{d}Y_t + U(X_t) \cdot \mathrm{d}t$ where $Y$ is a diffusion in $\mathbf{R}^d$ with generator $L$

To give the precise definition of a diffusion with drift $\mu$ in $\mathbf{K}_{d,1}$, we fix a non-negative smooth radial function $\varphi(x)$ in $\mathbf{R}^d$ with $\mathrm{supp}[\varphi] \subset B(0,1)$ and $\int \varphi(x)\mathrm{d}x = 1$. For any positive integer $n$, we put $\varphi_n(x) = 2^{nd}\varphi(2^n x)$. For $1 \le i \le d$, define

$$U_n^i(x) = \int \varphi_n(x-y)\mu^i(\mathrm{d}y).$$

Put $U_n(x) = (U_n^1(x), \dots, U_n^d(x))$.

In the definition below, we assume $a_{ij}$ is symmetric.

**Definition 1.2.** Suppose $\mu = (\mu^1, \dots, \mu^d)$ is such that each $\mu^i$ is a signed measure on $\mathbf{R}^d$ belonging to the Kato class $\mathbf{K}_{d,1}$. A diffusion with drift $\mu$ is a family of probability measures $\{\mathbf{P}_x : x \in \mathbf{R}^d\}$ on $C([0,\infty), \mathbf{R}^d)$, the space of continuous $\mathbf{R}^d$-valued functions on $[0,\infty)$, such that under each $\mathbf{P}_x$ we have

$$X_t = x + Y_t + A_t$$

where

(a) $A_t = \lim_{n\to\infty} \int_0^t U_n(X_s)\mathrm{d}s$ uniformly in $t$ over finite intervals, where the convergence is in probability;

(b) there exists a subsequence $\{n_k\}$ such that

$$\sup_k \int_0^t |U_{n_k}(X_s)|\mathrm{d}s < \infty$$

almost surely for each $t > 0$;

(c) $Y_t$ is a diffusion in $\mathbf{R}^d$ starting from the origin with generator $L$.

The existence and uniqueness of $X$ were established in [2] (see Remark 6.1 in [2]). For any open set $D$, we use $\tau_D$ to denote the first exit time of $D$, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbf{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \ge \tau_D(\omega)$, where $\partial$ is a cemetery state. The process $X^D$ is called a killed diffusion with drift $\mu$ in $D$. In this paper we discuss properties of $X^D$ when $D$ is a bounded domain.

When $a_{ij}$ is not symmetric, we use a simple reduction: Let $Y_t$ be a diffusion in $\mathbf{R}^d$ with generator

$$\frac{1}{4}\sum_{i,j=1}^d \partial_i((a_{ij} + a_{ji})\partial_j).$$

Note that

$$\sum_{i,j=1}^d a_{ij}\partial_i\partial_j = \sum_{i,j=1}^d \frac{1}{2}(a_{ij} + a_{ji})\partial_i\partial_j = \sum_{i,j=1}^d \frac{1}{2}\partial_i((a_{ij} + a_{ji})\partial_j) - \sum_{i,j=1}^d \frac{1}{2}\partial_i(a_{ij} + a_{ji})\partial_j$$

and

$$\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) = \sum_{i,j=1}^d a_{ij}\partial_i\partial_j + \sum_{i,j=1}^d (\partial_i a_{ij})\partial_j$$

$$= \sum_{i,j=1}^{d} \frac{1}{2}\partial_i((a_{ij} + a_{ji})\partial_j) + \sum_{i,j=1}^{d} \frac{1}{2}\partial_i(a_{ij} - a_{ji})\partial_j.$$

Since, for any bounded domain $D$,

$$\left(\sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{i1} + a_{1i})|_D, \ldots, \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{id} + a_{di})|_D\right)$$

and

$$\left(\sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{i1} - a_{1i})|_D, \ldots, \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{id} - a_{1d})|_D\right)$$

are in $\mathbf{K}_{d,1}$, we can construct $X_t$ with a drift which is either equal to

$$\left(\mu^1 + \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{i1} + a_{1i})|_D \mathrm{d}x, \ldots, \mu^d + \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{id} + a_{di})|_D \mathrm{d}x\right)$$

or equal to

$$\left(\mu^1 + \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{i1} - a_{1i})|_D \mathrm{d}x, \ldots, \mu^d + \sum_{i=1}^{d} \frac{1}{4}\partial_i(a_{id} - a_{1d})|_D \mathrm{d}x\right)$$

as in Definition 1.2. Then the generator of the killed diffusion process $X^D$ in $D$ can be formally written as $L + \mu \cdot \nabla$ where $L$ is either

$$\frac{1}{2}\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j) \quad \text{and} \quad \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j$$

with $\mathbf{A} := (a_{ij})$ being $C^1$ and uniformly elliptic but not necessarily symmetric.

Throughout this paper we assume that $X^D$ is the process constructed above. In [16] (also see Section 6 in [17]), we showed that $X$ has a density $q(t, x, y)$ which is continuous on $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$ and that there exist positive constants $c_i$, $i = 1, \ldots, 9$, such that

$$c_1 \mathrm{e}^{-c_2 t} t^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_3|x-y|^2}{2t}} \le q(t, x, y) \le c_4 \mathrm{e}^{c_5 t} t^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_6|x-y|^2}{2t}} \tag{1.1}$$

and

$$|\nabla_x q(t, x, y)| \le C_7 \mathrm{e}^{c_8 t} t^{-\frac{d+1}{2}} \mathrm{e}^{-\frac{c_9|x-y|^2}{2t}} \tag{1.2}$$

for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We also showed that, for every bounded $C^{1,1}$ domain $D$ (see below for the definition), $X^D$ has a density $q^D$ which is continuous on $(0, \infty) \times D \times D$ and that for any $T > 0$, there exist positive constants $c_i$, $i = 10, \ldots, 14$, such that

$$\begin{aligned} c_{10} t^{-\frac{d}{2}} \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \mathrm{e}^{-\frac{c_{11}|x-y|^2}{t}} &\le q^D(t, x, y) \\ \le c_{12} \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_{13}|x-y|^2}{t}} & \end{aligned} \tag{1.3}$$

and

$$|\nabla_x q^D(t,x,y)| \le c_{14}\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d+1}{2}} e^{-\frac{c_{13}|x-y|^2}{t}} \tag{1.4}$$

for all $(t,x,y) \in (0,T] \times D \times D$, where $a \wedge b := \min\{a,b\}$, $\rho(x)$ is the distance between $x$ and $\partial D$. Using the estimates above we studied the potential theory of $X$ in [16,17]. More precisely, we proved the boundary Harnack principle holds for non-negative harmonic functions of $X$ in bounded Lipschitz domains and identified the Martin boundary of the killed process $X^D$ when $D$ is a bounded Lipschitz domain.

In general, the process $X$ does not have a dual and therefore results for Markov processes under the duality hypotheses, like the general conditional gauge theorems of [6,9] or the stability of Martin boundary under perturbation of [8], cannot be applied to $X$ directly. The important concept of intrinsic ultracontractivity was introduced by Davies and Simon in [14] for symmetric semigroups and many people have made important contribution in studying the intrinsic ultracontractivity of symmetric semigroups. In [19] the concept of intrinsic ultracontractivity was extended to non-symmetric semigroups and it was proved there that the semigroup of a killed diffusion process in a bounded Lipschitz domain is intrinsic ultracontractive if the coefficients of the generator of the diffusion process are smooth. We would like to prove that, under very weak assumptions on the domain, the intrinsic ultracontractivity for the semigroups of killed diffusion processes with measure-valued drift. However, the existence of a dual process is crucial in establishing the intrinsic ultracontractivity. This is our main motivation for proving the existence of dual processes and for studying the properties of the dual processes.

In this paper we will first prove that, for any bounded domain $D$, $X^D$ has a dual process with respect to a certain reference measure and the dual process is a continuous Hunt process on $D$ with the strong Feller property. Then we study properties of the dual process. The main results of this paper are, besides the existence of the dual process, are (1) Theorem 4.11 which says that a scale invariant Harnack inequality is valid for the dual process; (2) Theorem 6.7 in which we prove that, if the domain is bounded $C^{1,1}$, the boundary Harnack principle for the dual process is valid; and (3) Theorem 6.6 in which we identify the (minimal) Martin boundary for the dual process. The key to establishing the results above is Theorem 4.7 which says that the harmonic measure for the dual process is locally comparable to that of the $h$-conditioned Brownian motion with $h$ being an eigenvalue corresponding to the largest Dirichlet eigenvalue in $D$. In [20] we will use the results of this paper to show that the Schrödinger semigroup of the killed process $X^D$ is intrinsic ultracontractive under very weak assumptions on $D$.

The content of this paper is organized as follows. In Section 2, we present some preliminary properties of the killed process $X^D$ in an arbitrary bounded domain $D$; the existence of the dual process of $X^D$ is proved in Section 3; Section 4 contains a result on the comparison of harmonic measures and a scale invariant Harnack inequality for the dual process which is used in Sections 5 and 6 to study the Martin boundary of the dual process; and in Section 7 we specialize the general conditional gauge theorems of [6,9] to the present setting and then, using the stability result of [8], get the stability of Martin boundaries of $X^D$ and its dual under perturbations.

Throughout this paper, we use the notation $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. We will use the convention $f(\partial) = 0$ for any function $f$ on $D$. In this paper we will also use the following convention: the values of the constants $c_1, c_2, \ldots$ might change from one appearance to another. The labeling of the constants $c_1, c_2, \ldots$ starts anew in the statement of each result. In this paper, we use “:=” to denote a definition, which is read as “is defined to be”.

## 2. Diffusion with measure-valued drift in bounded domains

In this section we assume that $D$ is an arbitrary bounded domain and we will discuss some basic properties of $X^D$ that we will need later.

It is shown in [17] that $X^D$ has a jointly continuous and strictly positive transition density function $q^D(t,x,y)$. Using the continuity $q^D(t,x,y)$ and the estimate (1.1), the proof of the next proposition is easy. We omit the proof.

**Proposition 2.1.** *$X^D$ is a Hunt processes and has the strong Feller property. i.e., for every $f \in L^\infty(D)$, $P_t^D f(x) := \mathbf{E}_x[f(X_t^D)]$ is bounded and continuous in $D$.*

We know from [17] that there exist positive constants $c_1$ and $c_2$ depending on $D$ via its diameter such that for any $(t,x,y) \in (0,\infty) \times D \times D$,

$$q^D(t,x,y) \le c_1 t^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_2|x-y|^2}{2t}}. \tag{2.1}$$

Let $G_D(x,y)$ be the Green function of $X^D$, i.e.,

$$G_D(x,y) := \int_0^\infty q^D(t,x,y)\mathrm{d}t.$$

$G_D(x,y)$ is finite for $x \neq y$ and

$$G_D(x,y) \le \frac{c_3}{|x-y|^{d-2}} \tag{2.2}$$

for some $c_3 = c_3(\mathrm{diam}(D)) > 0$. Now define

$$h_D(x) := \int_D G_D(y,x)\mathrm{d}y \quad \text{and} \quad \xi_D(\mathrm{d}x) := h_D(x)\mathrm{d}x.$$

The following result says that $\xi_D$ is a reference measure for $X^D$.

**Proposition 2.2.** *For any bounded domain $D$, $\xi_D$ is an excessive measure with respect to $X^D$, i.e., for every Borel function $f \ge 0$,*

$$\int_D f(x)\xi_D(\mathrm{d}x) \ge \int_D \mathbf{E}_x\left[f(X_t^D)\right]\xi_D(\mathrm{d}x).$$

*Moreover, $h_D$ is a strictly positive, bounded continuous function on $D$.*

**Proof.** By the Markov property, we have for any Borel function $f \ge 0$,

$$\begin{aligned}\int_D \mathbf{E}_y\left[f(X_t^D)\right]G_D(x,y)\mathrm{d}y &= \mathbf{E}_x \int_0^\infty \mathbf{E}_{X_s^D}\left[f(X_t^D)\right]\mathrm{d}s \\ &= \int_0^\infty \mathbf{E}_x\left[f(X_{t+s}^D)\right]\mathrm{d}s \le \int_D f(y)G_D(x,y)\mathrm{d}y, \quad x \in D.\end{aligned}$$

Integrating with respect to $x$, we get by Fubini's theorem,

$$\int_D \mathbf{E}_y\left[f(X_t^D)\right]h_D(y)\mathrm{d}y \le \int_D f(y)h_D(y)\mathrm{d}y.$$

The second claim follows from the continuity of $G_D$ and (2.2). $\square$

We define a new transition density function by

$$\overline{q}^D(t,x,y) := \frac{q^D(t,x,y)}{h_D(y)}.$$

Let

$$\overline{G}_D(x,y) := \int_0^\infty \overline{q}^D(t,x,y)\mathrm{d}t = \frac{G_D(x,y)}{h_D(y)}.$$

Then $\overline{G}_D(x,y)$ is the Green function of $X^D$ with respect to the reference measure $\xi_D$.

Before we discuss properties of $\overline{G}_D(x,y)$, let's first recall some definitions.

**Definition 2.3.** Suppose $U$ is an open subset of $D$. A Borel function $u$ defined on $U$ is said to be

(1) harmonic with respect to $X^D$ in $U$ if

$$u(x) = \mathbf{E}_x\left[u(X^D_{\tau_B})\right], \quad x \in B, \tag{2.3}$$

for every bounded open set $B$ with $\overline{B} \subset U$;

(2) superharmonic with respect to $X^D$ if

$$u(x) \geq \mathbf{E}_x\left[u(X^D_{\tau_B})\right], \quad x \in B,$$

for every bounded open set $B$ with $\overline{B} \subset D$;

(3) excessive for $X^D$ if $u$ is non-negative and

$$u(x) \geq \mathbf{E}_x\left[u(X^D_t)\right] \quad \text{and} \quad u(x) = \lim_{t\downarrow 0}\mathbf{E}_x\left[u(X^D_t)\right], \quad t>0, x\in D;$$

(4) a potential for $X^D$ if it is excessive for $X^D$ and for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = D$,

$$\lim_{n\to\infty}\mathbf{E}_x\left[u(X^D_{\tau_{U_n}})\right] = 0; \quad \xi_D\text{-a.e. } x \in D;$$

(5) a pure potential for $X^D$ if it is a potential for $X^D$ and

$$\lim_{t\to\infty}\mathbf{E}_x\left[u(X^D_t)\right] = 0, \qquad \xi_D\text{-a.e. } x \in D.$$

A Borel function $u$ defined on $\overline{U}$ is said to be regular harmonic with respect to $X^D$ in $U$ if $u$ is harmonic with respect to $X^D$ in $U$ and (2.3) is true for $B = U$;

A Borel function $u$ defined on $D$ is said to be harmonic with respect to $X^D$ if it is harmonic with respect to $X^D$ in $D$.

Since $X^D$ is a Hunt processes with the strong Feller property, it is easy to check that $u$ is excessive for $X^D$ if and only if $u$ is lower semi-continuous in $D$ and superharmonic with respect to $X^D$. (See Theorem 4.5.3 in [13] for the Brownian motion case, and the proof there can adapted easily to the present case.)

We list some properties of the Green function $\overline{G}_D(x,y)$ of $X^D$ that we will need later.

(A1) $\overline{G}_D(x,y) > 0$ for all $(x,y) \in D\times D$; $\overline{G}_D(x,y) = \infty$ if and only if $x = y \in D$;

(A2) For every $x \in D$, $\overline{G}_D(x,\cdot)$ and $\overline{G}_D(\cdot,x)$ are extended continuous in $D$;

(A3) For every compact subset $K$ of $D$, $\int_K \overline{G}_D(x, y)\xi_D(dy) < \infty$.

The above properties can be checked easily from Theorem 2.6 in [17] and our Proposition 2.2 and (2.2) above. Thus $X^D$ is a transient diffusion satisfying the conditions in [11,25]. Applying Theorem 1 in [25], we have that

(A4) for each $y, x \to \overline{G}_D(x, y)$ is excessive for $X^D$ and harmonic for $X^D$ in $D \setminus \{y\}$. Moreover, for every open subset $U$ of $D$, we have

$$\mathbf{E}_x[\overline{G}_D(X^D_{T_U}, y)] = \overline{G}_D(x, y), \quad (x, y) \in D \times U \tag{2.4}$$

where $T_U := \inf\{t > 0 : X^D_t \in U\}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $\overline{G}_D(\cdot, y)$ is regular harmonic in $D \setminus B(y, \varepsilon)$ with respect to $X^D$.

By combining (A4) above with Corollary 2 and Theorems 5–6 in [11], we get the following Riesz decomposition theorem for $X^D$.

**Theorem 2.4.** (1) *If $u$ is a potential for $X^D$, then there exists a unique Radon measure $\nu$ on $D$ such that*

$$u(x) = G_D\nu(x) := \int_D \overline{G}_D(x, y)\nu(dy).$$

(2) *If $f$ is an excessive function for $X^D$ and $f$ is not identically zero, then there exists a unique Radon measure $\nu$ on $D$ and a non-negative harmonic function $h$ for $X^D$ such that $f = G_D\nu + h$.*

For $y \in D$, let $X^{D,y}$ denote the $h$-conditioned process obtained from $X^D$ with $h(\cdot) = \overline{G}_D(\cdot, y)$ and let $\mathbf{E}^y_x$ denote the expectation for $X^{D,y}$ starting from $x \in D$.

**Theorem 2.5.** *For each $y \in D$, $x \to \overline{G}_D(x, y)$ is a pure potential for $X^D$. In fact, for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = D$,*

$$\lim_{n\to\infty} \mathbf{E}_x\left[\overline{G}_D(X^D_{\tau_{U_n}}, y)\right] = 0, \quad x \neq y.$$

*Moreover, for every $x, y \in D$, we have*

$$\lim_{t\to\infty} \mathbf{E}_x\left[\overline{G}_D(X^D_t, y)\right] = 0.$$

**Proof.** Let $x \neq y \in D$. Using (A1)–(A2), (A4) and the strict positivity of $\overline{G}_D$, and applying Theorem 2 in [23], we get that the lifetime $\zeta^y$ for $X^{D,y}$ is finite $\mathbf{P}^y_x$-a.s. and

$$\lim_{t\uparrow\zeta^y} X^{D,y}_t = y \qquad \mathbf{P}^y_x\text{-a.s.} \tag{2.5}$$

Let $\{D_k, k \geq 1\}$ be an increasing sequence of relatively compact open subsets of $D$ such that $D_k \subset \overline{D_k} \subset D$ and $\cup_{k=1}^\infty D_k = D$. Then

$$\mathbf{E}_x\left[\overline{G}_D(X^D_{\tau_{D_k}}, y)\right] = \overline{G}_D(x, y)\mathbf{P}^y_x(\tau_{D_k} < \zeta^y).$$

By (2.5), we have $\lim_{k\to\infty} \mathbf{P}^y_x(\tau_{D_k} < \zeta^y) = 0$. Thus

$$\lim_{k\to\infty} \mathbf{E}_x\left[\overline{G}_D(X^D_{\tau_{D_k}}, y)\right] = 0. \tag{2.6}$$

The last claim in the theorem is easy. By (2.1) and (2.2), for every $x, y \in D$, we have

$$\mathbf{E}_x\left[\overline{G}_D(X_t^D, y)\right] \le \frac{c}{t^{\frac{d}{2}} h_D(y)} \int_D \frac{\mathrm{d}z}{|z-y|^{d-2}},$$

which converges to zero as $t$ goes to $\infty$. □

The proof of the next proposition can be found in the proofs of Theorems 2–3 in [25]. Since we will refer to the argument of the proof of the proposition later, we include the proof here for the reader's convenience.

**Proposition 2.6.** *If $h$ is a non-negative harmonic function for $X^D$ and $U$ is an open subset of $D$ with $\overline{U} \subset D$, then there exists a Radon measure $\nu$ supported on $\partial U$ such that $h = G_D \nu$ in $U$. In particular, every non-negative harmonic function for $X^D$ is continuous.*

**Proof.** Using (2.2) and (A1)–(A2), and applying Corollary 1 in [26], we get that $h$ is excessive. Let $T_U := \inf\{t > 0 : X_t^D \in U\}$. Since $h$ is excessive, Corollary 1 to Theorem 2 in [11] implies that there exists a Radon measure $\nu$ supported on $\overline{U}$ such that $\mathbf{E}_x[h(X^D_{T_U})] = G_D\nu(x)$ for all $x \in D$. Since

$$G_D\nu(x) = \int_U \overline{G}_D(x, y)\nu(\mathrm{d}y) + \int_{\partial U} \overline{G}_D(x, y)\nu(\mathrm{d}y) =: h_1(x) + h_2(x), \quad x \in D$$

and $h_1$ and $h_2$ are excessive (Theorem 2.4), $h_1$ and $h_2$ must be harmonic with respect to $X^D$. Let $K$ be a compact subset of $U$. By the harmonicity of $h_1$, we have

$$\mathbf{E}_x[h_1(X^D_{T_{K^c}})] = \int_{\partial U} \overline{G}_D(x, y)\nu(\mathrm{d}y).$$

But, by Corollary 1 to Theorem 2 in [11], $\nu$ can not charge the interior of $K$. Since $K$ is an arbitrary compact subset of $U$, we get that $h_1$ is identically zero and $\nu$ is supported by $\partial U$. Therefore we have shown $h(x) = \mathbf{E}_x[h(X^D_{T_U})] = G_D\nu(x)$ for $x \in U$. Now the continuity of $h$ follows from the continuity of $\overline{G}_D(x, y)$. □

## 3. Dual of $X^D$ in bounded domains

In this section we assume that $D$ is an arbitrary bounded domain. First we show that $X^D$ has a nice dual process with respect to $\xi_D$ and then we will study the dual process of $X^D$.

Recall that $h_D(x) = \int_D G_D(y, x)\mathrm{d}y$, $\xi_D(\mathrm{d}x) = h_D(x)\mathrm{d}x$, $\overline{q}^D(t, x, y) = \frac{q^D(t,x,y)}{h_D(y)}$ and $\overline{G}_D(x, y) = \frac{G_D(x,y)}{h_D(y)}$. We note that

$$\int_D \overline{G}_D(x, y)\xi_D(\mathrm{d}x) \le \frac{\|h_D\|_{L_\infty(D)}}{h_D(y)} \int_D G_D(x, y)\mathrm{d}x = \|h_D\|_{L_\infty(D)} < \infty.$$

So we have

(A5) for every compact subset $K$ of $D$, $\int_K \overline{G}_D(x, y)\xi_D(\mathrm{d}x) < \infty$.

Using (A1)–(A5), (2.2) and Theorem 2.5 we get from [22] that $X^D$ has a continuous Hunt process as a dual process.

**Theorem 3.1.** *There exists a transient continuous Hunt process $\widehat{X}^D$ in $D$ such that $\widehat{X}^D$ is a strong dual of $X^D$ with respect to the measure $\xi_D$, that is, the density of the semigroup $\{\widehat{P}_t^D\}_{t\geq 0}$ of $\widehat{X}^D$ is given by $\widehat{q}^D(t,x,y) := \overline{q}^D(t,y,x)$ and thus*

$$\int_D f(x) P_t^D g(x) \xi_D(\mathrm{d}x) = \int_D g(x) \widehat{P}_t^D f(x) \xi_D(\mathrm{d}x) \quad \text{for all } f, g \in L^2(D, \xi_D).$$

**Proof.** The existence of a dual continuous Hunt process $\widehat{X}^D$ is proved in [22]. To show $\widehat{X}^D$ is transient, we need to show that for every compact subset $K$ of $D$, $\int_K \overline{G}_D(x,y)\xi_D(\mathrm{d}x)$ is bounded. This is just (A5) above. □

We will use $\widehat{\zeta}$ to denote the lifetime of $\widehat{X}^D$. Note that the process $\widehat{X}^D$ might be killed inside $D$, that is, the probability of the event $\widehat{X}^D_{\widehat{\zeta}-} \in D$ might be strictly positive.

Using (A1)–(A2) and (A5), and applying Theorem 1 in [25] to $\widehat{X}^D$ we get that

(A6) for each $y$, $x \to \overline{G}_D(y,x)$ is excessive for $\widehat{X}^D$ and harmonic in $D \setminus \{y\}$. Moreover, for every open subset $U$ of $D$, we have

$$\mathbf{E}_x\left[\overline{G}_D(y, \widehat{X}^D_{\widehat{T}_U})\right] = \overline{G}_D(y,x), \quad (x,y) \in D \times U \tag{3.1}$$

where $\widehat{T}_U := \inf\{t > 0 : \widehat{X}_t^D \in U\}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $\overline{G}_D(y,\cdot)$ is regular harmonic in $D \setminus B(y,\varepsilon)$ with respect to $\widehat{X}^D$.

Thus the Riesz decomposition theorem (Theorem 2.4) is valid for $\widehat{X}^D$ too.

**Theorem 3.2.** (1) *If $u$ is a potential for $\widehat{X}^D$, then there exists a unique Radon measure $\nu$ on $D$ such that*

$$u(x) = \widehat{G}_D \nu(x) := \int_D \overline{G}_D(y,x)\nu(\mathrm{d}y)$$

(2) *If $f$ is an excessive function for $\widehat{X}^D$ and $f$ is not identically zero, then there exists a unique Radon measure $\nu$ on $D$ and a non-negative harmonic function $h$ for $\widehat{X}^D$ such that $f = \widehat{G}_D\nu + h$.*

**Theorem 3.3.** *For each $y$, $x \to \overline{G}_D(y,x)$ is a pure potential for $\widehat{X}^D$. In fact, for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = D$,*

$$\lim_{n\to\infty} \mathbf{E}_x\left[\overline{G}_D(y, \widehat{X}^D_{\widehat{\tau}_{U_n}}); \widehat{\tau}_{U_n} < \widehat{\zeta}\right] = 0, \quad x \neq y.$$

*Moreover, for every $x, y \in D$, we have*

$$\lim_{t\to\infty} \mathbf{E}_x\left[\overline{G}_D(y, \widehat{X}_t^D); t < \widehat{\zeta}\right] = 0.$$

**Proof.** The first assertion can be proved using the same argument as in the proof of Theorem 2.5, so we only need to prove the last assertion.

By (2.1) and (2.2), we have for every $x, y \in D$,

$$\mathbf{E}_x\left[\overline{G}_D(y, \widehat{X}_t^D); t < \widehat{\zeta}\right] = \int_D \frac{q^D(t,z,x)}{h_D(x)} \overline{G}_D(y,z) h_D(z)\mathrm{d}z \leq \frac{c}{t^{\frac{d}{2}} h_D(x)} \int_D \frac{\mathrm{d}z}{|z-y|^{d-2}},$$

which converges to zero as $t$ goes to $\infty$. □

Note that every non-negative harmonic function for $\widehat{X}^D$ is excessive and continuous by Corollary 1 in [26]. For any $\alpha \geq 0$, put

$$\widehat{G}^D_\alpha(x,y) := \int_0^\infty \mathrm{e}^{-\alpha t}\widehat{q}^D(t,x,y)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\alpha t}\overline{q}^D(t,y,x)\mathrm{d}t.$$

**Proposition 3.4.** *$\widehat{X}^D$ has the strong Feller property in the resolvent sense; that is, for every bounded Borel function $f$ on $D$ and $\alpha \geq 0$, $\widehat{G}^D_\alpha f(x)$ is a bounded continuous function on $D$.*

**Proof.** By the resolvent equation $\widehat{G}^D_0 = \widehat{G}^D_\alpha + \alpha\widehat{G}^D_0\widehat{G}^D_\alpha$, it is enough to show the strong Feller property for $\widehat{G}^D_0$. Fix a bounded Borel function $f$ on $D$ and a sequence $\{y_n\}_{n\geq 1}$ converges to $y$ in $D$. Let $M := \|fh_D\|_{L_\infty(D)} < \infty$. We assume $\{y_n\}_{n\geq 1} \subset K$ for a compact subset $K$ of $D$. Let $A := \inf_{y\in K} h_D(y)$. By Proposition 2.2, we know that $A$ is strictly positive. Note that there exists a constant $c_1$ such that for every $\delta > 0$

$$\left(\int_{B(y,\delta)} \frac{\mathrm{d}x}{|x-y|^{d-2}} + \int_{B(y_n,2\delta)} \frac{\mathrm{d}x}{|x-y_n|^{d-2}}\right) \leq c_1\delta^2.$$

Thus by (2.2), there exists a constant $c_2$ such that for every $\delta > 0$ and $y_n$ with $y_n \in B(y, \frac{\delta}{2}) \subset B(y, 2\delta) \in K$,

$$\begin{aligned}
&\int_{B(y,\delta)} \overline{G}_D(x,y)f(x)\xi_D(\mathrm{d}x) + \int_{B(y,\delta)} \overline{G}_D(x,y_n)f(x)\xi_D(\mathrm{d}x)\\
&\quad\leq \frac{M}{A}\left(\int_{B(y,\delta)} G_D(x,y)\mathrm{d}x + \int_{B(y_n,2\delta)} G_D(x,y_n)\mathrm{d}x\right)\\
&\quad\leq \frac{c_2M}{A}\left(\int_{B(y,\delta)} \frac{\mathrm{d}x}{|x-y|^{d-2}} + \int_{B(y_n,2\delta)} \frac{\mathrm{d}x}{|x-y_n|^{d-2}}\right) \leq \frac{1}{A}c_1c_2M\delta^2.
\end{aligned}$$

Given $\varepsilon$, choose $\delta$ small enough such that $\frac{1}{A}c_1c_2M\delta^2 < \frac{\varepsilon}{2}$. Then

$$|\widehat{G}^D_0 f(y) - \widehat{G}^D_0 f(y_n)| \leq M\int_{D\setminus B(y,\delta)} |\overline{G}_D(x,y) - \overline{G}_D(x,y_n)|\mathrm{d}x + \frac{\varepsilon}{2}.$$

Note that $\overline{G}_D(x,y_n)$ converges to $\overline{G}_D(x,y)$ for every $x \neq y$ and $\{\overline{G}_D(x,y_n)\}_{n\geq 1}$ are uniformly bounded on $x \in D \setminus B(y,\delta)$ and $y_n \in B(y, \frac{\delta}{2})$. So the first term on the right hand side of the inequality above goes to zero as $n \to \infty$ by the bounded convergence theorem. □

## 4. Comparison of harmonic measures and scale invariant Harnack inequality for the dual process

In this section we still assume that $D$ is an arbitrary bounded domain. For any open subset $U$ of $D$, we use $\widehat{X}^{D,U}$ to denote the process obtained by killing $\widehat{X}^D$ upon exiting $U$, i.e., $\widehat{X}^{D,U}_t(\omega) = \widehat{X}^D_t(\omega)$ if $t < \widehat{\tau}^D_U(\omega)$ and $\widehat{X}^{D,U}_t(\omega) = \partial$ if $t \geq \widehat{\tau}^D_U(\omega)$, where $\widehat{\tau}^D_U := \inf\{t > 0 : \widehat{X}^D_t \notin U\}$ and $\partial$ is the cemetery state. Then by Theorem 2 and Remark 2 after it in [29], $X^U$ and $\widehat{X}^{D,U}$ are dual processes with respect to $\xi_D$. Now we let

$$\widehat{q}^{D,U}(t,x,y) := \frac{q^U(t,y,x)h_D(y)}{h_D(x)}.$$

By the joint continuity of $q^U(t, x, y)$ (Theorem 2.4 in [16]) and the continuity and positivity of $h_D$ (Proposition 2.2), we know that $\widehat{q}^{D,U}(t, \cdot, \cdot)$ is jointly continuous on $U \times U$. Thus we have the following.

**Theorem 4.1.** *For every open subset $U$, $\widehat{q}^{D,U}(t, x, y)$ is jointly continuous on $U \times U$ and is the transition density of $\widehat{X}^{D,U}$ with respect to the Lebesgue measure. Moreover,*

$$\widehat{G}_{D,U}(x, y) := \frac{G_U(y, x)h_D(y)}{h_D(x)} \tag{4.1}$$

*is the Green function of $\widehat{X}^{D,U}$ with respect to the Lebesgue measure so that for every non-negative Borel function $f$,*

$$\mathbf{E}_x\left[\int_0^{\widehat{\tau}_U^D} f\left(\widehat{X}_t^D\right)\mathrm{d}t\right] = \int_U \widehat{G}_{D,U}(x, y)f(y)\mathrm{d}y.$$

Using (4.1) and repeating the argument in the proof of Proposition 2.6, we get the following.

**Proposition 4.2.** *If $h$ is a non-negative harmonic for $\widehat{X}^D$ in $U$ and $V$ is an open subset of $U$ with $\overline{V} \subset D$, then there exists a Radon measure $\nu$ supported on $\partial V$ such that*

$$h(x) = \int_{\partial V} \frac{G_U(y, x)h_D(y)}{h_D(x)}\nu(\mathrm{d}y), \quad x \in V.$$

*In particular, every non-negative harmonic function for $\widehat{X}^D$ in $U$ is continuous.*

Using (1.1) and Proposition 2.2, we see that for every compact subset $K$ of $D$, there exist $c_1$, $c_2$ and $c_3$ such that for every positive $t_0$ and $\delta$,

$$\begin{aligned}\sup_{t\le t_0, x\in K, |x-y|>\delta} \frac{q^D(t, y, x)h_D(y)}{h_D(x)} &\le c_1\mathrm{e}^{c_2 t_0} \sup_{t\le t_0, x\in K, |x-y|>\delta} t^{-\frac{d}{2}}\mathrm{e}^{-c_3\frac{|x-y|^2}{t}} \\ &\le c_1\mathrm{e}^{c_2 t_0} \sup_{t\le t_0} t^{-\frac{d}{2}}\mathrm{e}^{-c_3\frac{\delta^2}{t}} < \infty. \end{aligned} \tag{4.2}$$

(4.2) implies that for any compact subset $K$ of $D$,

$$\begin{aligned}\sup_{t\le t_0, x\in K} \mathbf{P}_x(|\widehat{X}_t^D - x| \ge \delta; t < \widehat{\zeta}) &\le c_1\mathrm{e}^{c_2 t_0} \sup_{t\le t_0, x\in K} \int_{|x-y|\ge\delta} t^{-\frac{d}{2}}\mathrm{e}^{-c_3\frac{|x-y|^2}{t}}\mathrm{d}y \\ &= c_4\mathrm{e}^{c_2 t_0} \sup_{t\le t_0} \int_\delta^\infty t^{-\frac{d}{2}} r^{d-1}\mathrm{e}^{-c_3\frac{r^2}{t}}\mathrm{d}r \\ &\le c_5\mathrm{e}^{c_2 t_0} \int_{\frac{\delta}{\sqrt{t_0}}}^\infty u^{d-1}\mathrm{e}^{-c_3 u^2}\mathrm{d}u \end{aligned}$$

for some $c_4 = c_4(d)$ and $c_5 = c_5(d)$. Thus

$$\lim_{t_0\downarrow 0} \sup_{t\le t_0, x\in K} \mathbf{P}_x(|\widehat{X}_t^D - x| \ge \delta; t < \widehat{\zeta}) = \lim_{t_0\downarrow 0} \sup_{t\le t_0, x\in K} \mathbf{P}_x(\widehat{X}_t^D \in D \setminus B(x, \delta)) = 0. \tag{4.3}$$

Using (4.3) we can easily prove the next lemma.

**Lemma 4.3.** *For any $\delta > 0$ and $x \in D$ with $B(x, 2\delta) \subset D$, we have*

$$\lim_{s\downarrow 0} \sup_{x\in D: B(x,2\delta)\in D} \mathbf{P}_x(\widehat{\tau}^D_{B(x,\delta)} \le s \wedge \widehat{\zeta}) = 0.$$

**Proof.** For any $t > 0$ and any Borel set $A$ in $D$, we put $N_t(x, A) = \mathbf{P}_x(\widehat{X}^D_t \in A)$. Then by an extended version of the strong Markov property (see page 43–44 of [3]), we have for every $x \in D$ with $B(x, 2\delta) \in D$,

$$\begin{aligned}
\mathbf{P}_x(\widehat{\tau}^D_{B(x,\delta)} \le s \wedge \widehat{\zeta}) &\le \mathbf{P}_x\left(\widehat{\tau}_{B(x,\delta)} \le s, \widehat{X}_s \in B\left(x, \frac{\delta}{2}\right)\right) \\
&\quad + \mathbf{P}_x\left(\widehat{X}^D_s \in B\left(x, \frac{\delta}{2}\right)^c, s < \widehat{\zeta}\right) \\
&\le \mathbf{E}_x\left[N_{s-\widehat{\tau}^D_{B(x,\delta)}}\left(\widehat{X}^D_{\widehat{\tau}_{B(x,\delta)}}, B\left(x, \frac{\delta}{2}\right)\right); \widehat{\tau}^D_{B(x,\delta)} \le s\right] \\
&\quad + \mathbf{P}_x\left(\widehat{X}^D_s \in B\left(x, \frac{\delta}{2}\right)^c, s < \widehat{\zeta}\right).
\end{aligned}$$

Since $X^D_{\tau_{B(x,\delta)}} \in \partial B(x, \delta)$ almost surely on $\{\widehat{\tau}_{B(x,\delta)} < \widehat{\zeta}\}$, the conclusion of the lemma follows from (4.3). □

A bounded domain $U$ in $\mathbf{R}^d$ is said to be a $C^{1,1}$ domain if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial U$, there is a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying $\phi(0) = |\nabla\phi(0)| = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \ldots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y : y_d > \phi(\tilde{y})\}$.

Using (1.3), it is easy to show the following.

**Proposition 4.4.** *For any bounded $C^{1,1}$ domain $U \subset D$ with $\overline{U} \subset D$, $\widehat{X}^{D,U}$ satisfies the strong Feller property in the semigroup sense; that is, for every bounded Borel function $f$ on $U$,*

$$\mathbf{E}_x\left[f(\widehat{X}^D_t); t < \widehat{\tau}^D_U\right]$$

*is a bounded continuous function on $U$.*

**Proof.** Fix $x_0 \in U$ and $t > 0$. Suppose $x_n \in U$ converges to $x_0 \in U$. Let $N := \inf_{n\ge 1} h_D(x_n) > 0$. Then

$$\begin{aligned}
&\left|\mathbf{E}_{x_n}\left[f(\widehat{X}^D_t); t < \widehat{\tau}^D_U\right] - \mathbf{E}_{x_0}\left[f(\widehat{X}^D_t); t < \widehat{\tau}^D_U\right]\right| \\
&\quad \le c_1 \int_U \left|\frac{q^U(t, y, x_n)}{h_D(x_n)} - \frac{q^U(t, y, x_0)}{h_D(x_0)}\right| h_D(y)\mathrm{d}y \\
&\quad \le c_2 \int_U \left(\left|\frac{q^U(t, y, x_n)}{h_D(x_n)} - \frac{q^U(t, y, x_0)}{h_D(x_n)}\right| + q^U(t, y, x_0)\left|\frac{1}{h_D(x_n)} - \frac{1}{h_D(x_0)}\right|\right)\mathrm{d}y \\
&\quad \le \frac{c_2}{N}\int_U \left|q^U(t, y, x_n) - q^U(t, y, x_0)\right|\mathrm{d}y + c_2\left|\frac{1}{h_D(x_n)} - \frac{1}{h_D(x_0)}\right|.
\end{aligned}$$

Given $\varepsilon > 0$, choose $n_0 > 0$ such that

$$c_2\left|\frac{1}{h_D(x_n)} - \frac{1}{h_D(x_0)}\right| < \frac{\varepsilon}{4}, \quad n \ge n_0.$$

Let $\rho_U(y)$ be the distance between $y$ and $\partial U$. By (1.3),

$$\begin{aligned}
&\int_U \left| q^U(t,y,x_n) - q^U(t,y,x_0)\right| \mathrm{d}y \\
&\quad = \left(\int_{\{y\in U:\rho_U(y)<\delta\}} + \int_{\{y\in U:\rho_U(y)\ge\delta\}}\right) \left| q^U(t,y,x_n) - q^U(t,y,x_0)\right| \mathrm{d}y \\
&\quad \le c_3\,|U|\delta t^{-\frac{d+1}{2}} + \int_{\{y\in U:\rho_U(y)\ge\delta\}} \left| q^U(t,y,x_n) - q^U(t,y,x_0)\right| \mathrm{d}y
\end{aligned}$$

for some $c_3$. Now we choose $\delta$ small so that $c_2c_3|U|N^{-1}\delta t^{-\frac{d+1}{2}} < \frac{\varepsilon}{4}$. The convergence of the second term on the right hand side of the inequality above follows from the uniform continuity of $q^U(t,\cdot,\cdot)$ on $B(x_0, \frac{1}{2}\rho_U(x_0)) \times \{y \in U : \rho_U(y) \ge \delta\}$ (Theorem 3.1 in [16]). Thus we have proved the proposition. □

Recall that, a point $x$ on the boundary $\partial U$ of an open subset $U$ of $D$ is said to be a regular boundary point for $\widehat{X}^D$ in $U$ if $\mathbf{P}_x(\widehat{\tau}^D_U = 0) = 1$. An open subset $U$ of $D$ is said to be regular if every point in $\partial U$ is a regular boundary point.

**Proposition 4.5.** *Suppose $U$ is an open subset of $D$ with $\overline{U} \subset D$, and $z \in \partial U$. If there is a cone $A$ with vertex $z$ such that $A \cap B(z,r) \subset D \setminus U$ for some $r > 0$, then $z$ is a regular boundary point of $U$.*

**Proof.** Choose a bounded smooth domain $D_1$ with $\overline{U} \subset D_1 \subset \overline{D_1} \subset D$. Without loss of generality, we may assume that $z = 0$ and $A \cap B(z,r) \subset D_1 \setminus U$. For $n \ge 1$, put $r_n = r/n$. Under $\mathbf{P}_0$, we have

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\widehat{X}^D_{r_n} \in A \cap B(0,r)\} \subset \{\widehat{\tau}^D_U = 0\}.$$

Moreover, since $D_1$ is bounded smooth and $\overline{D_1} \subset D$, by (1.3) there exists a constant $c_1 > 0$ such that for $x \in A \cap B(0,r)$ and large $n$

$$\frac{q^{D_1}(t,x,0)h_D(x)}{h_D(0)} \ge c_1 r_n^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_2|x|^2}{r_n}}.$$

Hence

$$\begin{aligned}
\mathbf{P}_0(\widehat{\tau}^D_U = 0) &\ge \mathbf{P}_0\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\widehat{X}^D_{r_n} \in A \cap B(0,r)\}\right) \ge \limsup_{n\to\infty} \mathbf{P}_0(\widehat{X}^D_{r_n} \in A \cap B(0,r)) \\
&\ge \limsup_{n\to\infty} \mathbf{P}_0(\widehat{X}^{D_1}_{r_n} \in A \cap B(0,r)) = \limsup_{n\to\infty} \int_{A\cap B(0,r)} \frac{q^{D_1}(t,x,0)h_D(x)}{h_D(0)} \mathrm{d}x \\
&\ge \limsup_{n\to\infty} c_1 \int_{A\cap B(0,r)} r_n^{-\frac{d}{2}} \mathrm{e}^{-\frac{c_2|x|^2}{r_n}} \mathrm{d}x \ge \limsup_{n\to\infty} c_1 \int_{A\cap B(0,n)} \mathrm{e}^{-c_2|y|^2} \mathrm{d}y > 0.
\end{aligned}$$

The assertion of the proposition now follows from Blumenthal's zero–one law (Proposition I.5.17 in [3]). □

This result implies that all bounded Lipschitz domains (see below for the definition) are regular if their closures are in $D$. Modifying the argument in the second part of the proof of Theorem 1.23 in [12], we get the following result.

**Proposition 4.6.** *Suppose $U$ is an open set subset of $D$ with $\overline{U} \subset D$ and $f$ is a bounded Borel function on $\partial U$. If $z$ is a regular boundary point of $U$ for $\widehat{X}^D$ and $f$ is continuous at $z$*

$$\lim_{\overline{U}\ni x\to z} \mathbf{E}_x \left[ f\left(\widehat{X}^D_{\widehat{\tau}^D_U}\right); \widehat{\tau}^D_U < \widehat{\zeta} \right] = f(z).$$

**Proof.** With Lemma 4.3 and Proposition 4.4 in hand, the proof is routine. We omit the details. □

Let $G^0_D$ be the Green function of a Brownian motion $W$ in $D$. By Theorem 3.7 in [17], there exist constants $r_1 = r_1(d, \mu) > 0$ and $M_1 = M_1(d, \mu) > 1$ depending on $\mu$ only via the rate at which $\max_{1\le i\le d} M_{\mu^i}(r)$ goes to zero such that for $r \le r_1$, $z \in \mathbf{R}^d$, $x, y \in B(z, r)$,

$$M_1^{-1} G^0_{B(z,r)}(x, y) \le G_{B(z,r)}(y, x) \le M_1 G^0_{B(z,r)}(x, y). \tag{4.4}$$

We will fix the constants $r_1 > 0$ and $M_1 > 0$ above in the remainder of this section.

**Theorem 4.7.** *For any bounded domain $D$, $r \le r_1$, $z \in D$ and $x \in B^z_r := B(z, r) \subset \overline{B(z, r)} \subset D$, we have*

$$\begin{aligned} M_1^{-2} h_D(y)\mathbf{P}_x\left(W_{\tau_{B^z_r}} \in \mathrm{d}y\right) &\le h_D(x)\mathbf{P}_x\left(\widehat{X}^D_{\widehat{\tau}_{B^z_r}} \in \mathrm{d}y, \widehat{\tau}_{B^z_r} < \widehat{\zeta}\right) \\ &\le M_1^2 h_D(y)\mathbf{P}_x\left(W_{\tau_{B^z_r}} \in \mathrm{d}y\right). \end{aligned} \tag{4.5}$$

**Proof.** We fix $z_0 \in D$ and $r \le r_1$ with $B(z_0, r) \subset \overline{B(z_0, r)} \subset D$, and let $B := B(z_0, r)$. The idea of the proof is similar to that of Theorem 2.2 in [10]. We include the details here for the reader's convenience. Let $\varphi \ge 0$ is a continuous function on $\partial B$ and let

$$u(x) := \mathbf{E}_x[\varphi(\widehat{X}^D_{\widehat{\tau}^D_B}); \widehat{\tau}^D_B < \widehat{\zeta}].$$

By Proposition 4.6, we know that $u$ is harmonic for $\widehat{X}^D$ in $B$ and continuous on $\overline{B}$. Let $B(n) := B(z_0, (1 - \frac{1}{n})r)$, $T_n := \inf\{t > 0 : \widehat{X}^D_t \in B(n)\}$ and $u_n(x) := \mathbf{E}_x[u(\widehat{X}^{D,B}_{T_n})]$. Then by Proposition 4.2, there exist Radon measures $\nu_n$ supported on $\partial B(n)$ such that

$$u_n(x) = \frac{1}{h_D(x)} \int_{\partial B(n)} G_B(y, x)\nu_n(\mathrm{d}y).$$

Let

$$v_n(x) := \int_{\partial B(n)} G^0_B(x, y)\nu_n(\mathrm{d}y).$$

Then by (4.4),

$$M_1^{-1} v_n(x) \le h_D(x)u_n(x) \le M_1 v_n(x), \quad x \in B(n).$$

Since $v_n$ is a harmonic function in $B(m)$ for $m \ge n$, by the Hölder continuity of $v_n$ and a diagonalization procedure, there is a subsequence $n_k$ such that $v_{n_k}$ converges uniformly on each $B(m)$ to a harmonic function $v$ in $B$. Thus

$$M_1^{-1} v(x) \le h_D(x)u(x) \le M_1 v(x), \quad x \in B. \tag{4.6}$$

Since $B$ is regular for $\widehat{X}^D$ (Proposition 4.5), by taking the limit above and using Proposition 4.6, we get for every $w \in \partial B$

$$M_1^{-1} h_D(w)\varphi(w) \le \liminf_{B\ni x\to w} v(x) \le \limsup_{B\ni x\to w} v(x) \le M_1 h_D(w)\varphi(w). \tag{4.7}$$

Let

$$w(x) = \mathbf{E}_x\left[h_D(W_{\tau_B})\varphi(W_{\tau_B})\right].$$

$w$ is a harmonic function in $B$ and continuous in $\overline{B}$ with the boundary value $h_D(w)\varphi(w)$. Thus by the maximum principle and (4.7), we get $M_1^{-1}w(x) \le v(x) \le M_1 w(x)$. So by (4.6) $M_1^{-2}w(x) \le h_D(x)u(x) \le M_1^2 w(x)$, which is

$$\begin{aligned} M_1^{-2}\int_{\partial B}\varphi(w)h_D(w)\mathbf{P}_x\left(W_{\tau_B}\in dw\right) &\le \int_{\partial B}\varphi(w)h_D(x)\mathbf{P}_x\left(\widehat{X}^D_{\widehat{\tau}^D_B}\in dw, \widehat{\tau}^D_B<\widehat{\zeta}\right)\\ &\le M_1^2\int_{\partial B}\varphi(w)h_D(w)\mathbf{P}_x\left(W_{\tau_B}\in dw\right). \quad \square \end{aligned}$$

Recall that $\rho_D(x)$ is the distance between $x$ and $\partial D$.

**Lemma 4.8.** *Suppose $D$ is a bounded $C^{1,1}$ domain. Then there exists a constant $c = c(D)$ such that*

$$\frac{1}{c}\rho_D(x) \le h_D(x) \le c\rho_D(x). \tag{4.8}$$

**Proof.** Since $D$ is bounded, the Green function estimates for $X^D$ ((6.2)–(6.3) in [16]) imply that

$$c_1\rho_D(x)\left(1\wedge\frac{\rho_D(y)}{|x-y|^2}\right)\frac{1}{|x-y|^{d-2}} \le G_D(y,x) \le c_2\frac{\rho_D(x)}{|x-y|^{d-1}}$$

for some positive constants $c_1$ and $c_2$. Integrating over $y$ we get

$$c_1\rho_D(x)q_1(x) \le h_D(x) \le c_2\rho_D(x)q_2(x)$$

where

$$q_1(x) := \int_D\left(1\wedge\frac{\rho_D(y)}{|x-y|^2}\right)\frac{1}{|x-y|^{d-2}}\mathrm{d}y \quad \text{and} \quad q_2(x) := \int_D\frac{1}{|x-y|^{d-1}}\mathrm{d}y.$$

By elementary calculus, we easily see that $\inf_{x\in D} q_1(x) > 0$ and $\sup_{x\in D} q_2(x) < \infty$. $\square$

Let $\psi_0$ be the ground state for the killed Brownian motion in $D$, that is, $\psi_0$ is the positive eigenfunction corresponding to the largest eigenvalue of the Dirichlet Laplacian $\frac{1}{2}\Delta|_D$ with $\int_D \psi_0^2(\mathrm{d}x)\mathrm{d}x = 1$. If $D$ is bounded $C^{1,1}$, it is well known that there exists $c_1$ such that $c_1^{-1}\rho_D(x) \le \psi_0(x) \le c_1\rho_D(x)$. So we get the next result as a corollary of Theorem 4.7 and Lemma 4.8.

**Corollary 4.9.** *For any bounded $C^{1,1}$ domain $D$, there exists a positive constant $c$ such that for every $r \le r_1$, $z \in D$ and $x \in B^z_r := B(z,r) \subset \overline{B(z,r)} \subset D$, we have*

$$c^{-1}\mathbf{P}_x\left(W^{\psi_0}_{\tau_{B^z_r}}\in \mathrm{d}y\right) \le \mathbf{P}_x\left(\widehat{X}^D_{\widehat{\tau}_{B^z_r}}\in \mathrm{d}y, \widehat{\tau}_{B^z_r}<\widehat{\zeta}\right) \le c\,\mathbf{P}_x\left(W^{\psi_0}_{\tau_{B^z_r}}\in \mathrm{d}y\right) \tag{4.9}$$

*where $W^{\psi_0}$ is the $h$-conditioned Brownian motion with $h = \psi_0$.*

In the remainder of this section, we will prove a scale invariant Harnack inequality for $\widehat{X}^D$ for any bounded domain $D$. First we prove the following lemma. Recall that $r_1 > 0$ and $M_1 > 0$ are the constants from (4.4).

**Lemma 4.10.** *There exists a constant $c = c(D, \mu) > 1$ such that for every $r < r_1$ and $B(z, r) \subset D$,*

$$\frac{h_D(x)}{h_D(y)} \le c, \quad x, y \in B\left(z, \frac{r}{2}\right).$$

**Proof.** Since $r < r_1$, by (2.2) and (4.4), there exists $c_1 = c_1(D, \mu) > 1$ such that for every $x, w \in B(z, \frac{3r}{4})$

$$c_1^{-1} \frac{1}{|w - x|^{d-2}} \le M_1^{-1} G^0_{B(z,r)}(w, x) \le G_{B(z,r)}(w, x) \le G_D(w, x) \le c_1 \frac{1}{|w - x|^{d-2}}.$$

Thus for $w \in \partial B(z, \frac{3r}{4})$ and $x, y \in B(z, \frac{r}{2})$, we have

$$G_D(w, x) \le c_1 \left(\frac{|w - y|}{|w - x|}\right)^{d-2} \frac{1}{|w - y|^{d-2}} \le 4^{d-2} c_1^2 G_D(w, y). \tag{4.10}$$

On the other hand, from (2.4), we have

$$h_D(x) = \int_D G_D(a, x) \mathrm{d}a = \int_D \mathbf{E}_a \left[G_D(X^D_{T_{B\left(z, \frac{3r}{4}\right)}}, x)\right] \mathrm{d}a, \quad x \in B\left(z, \frac{3r}{4}\right). \tag{4.11}$$

Since $X^D_{T_{B(z, \frac{3r}{4})}} \in \partial B(z, \frac{3r}{4})$, combining (4.10) and (4.11), we get

$$h_D(x) \le 4^{d-2} c_1^2 \int_D \mathbf{E}_a \left[G_D(X_{T_{B\left(z, \frac{3r}{4}\right)}}, y)\right] \mathrm{d}a = 4^{d-2} c_1^2 h_D(y), \quad x, y \in B\left(z, \frac{r}{2}\right). \quad \square$$

Now we are ready to prove the scale invariant Harnack inequality.

**Theorem 4.11** (*Scale Invariant Harnack Inequality*). *There exists $N = N(d, \mu) > 0$ such that for every harmonic function $f$ of $\widehat{X}^D$ in $B(z_0, r)$ with $r \in (0, r_1]$ and $B(z_0, r) \subset D$, we have*

$$\sup_{y \in B(z_0, r/4)} f(y) \le N \inf_{y \in B(z_0, r/4)} f(y).$$

**Proof.** We fix $z_0 \in D$ and $r \le r_1$ with $B(z_0, r) \subset D$, and a harmonic function $f$ of $\widehat{X}^D$ in $B(z_0, r)$. By the harmonicity of $f$ and Theorem 4.7, for every $x \in B(z_0, \frac{r}{2})$,

$$\begin{aligned} f(x) &= \mathbf{E}_x \left[f\left(\widehat{X}^D_{\widehat{\tau}_{B\left(z_0, \frac{r}{2}\right)}}\right); \widehat{\tau}_{B\left(z_0, \frac{r}{2}\right)} < \widehat{\zeta}\right] \\ &\le \frac{M_1^2}{h_D(x)} \mathbf{E}_x \left[h_D\left(W_{\tau_{B\left(z_0, \frac{r}{2}\right)}}\right) f\left(W_{\tau_{B\left(z_0, \frac{r}{2}\right)}}\right)\right]. \end{aligned}$$

Thus by Lemma 4.10,

$$f(x) \le c M_1^2 \mathbf{E}_x \left[f\left(W_{\tau_{B(z_0, \frac{r}{2})}}\right)\right] =: c M_1^2 g(x)$$

for some constant $c$. Since $g$ is harmonic for $W$ in $B(z_0, \frac{r}{2})$, by the Harnack inequality for Brownian motion (for example, see [1]),

$$\frac{1}{c_1} g(y) \le g(x) \le c_1 g(y), \quad x, y \in B\left(z_0, \frac{r}{4}\right)$$

for some constant $c_1 > 0$. Thus by applying Theorem 4.7 and Lemma 4.10 again, we have that for every $x, y \in B(z_0, \frac{r}{4})$,

$$\begin{aligned} f(x) &\le c c_1 M_1^2 \mathbf{E}_y \left[ f\left( W_{\tau_{B(z_0, \frac{r}{2})}} \right) \right] \le c^2 c_1 M_1^4 \mathbf{E}_x \left[ f\left( \widehat{X}^D_{\widehat{\tau}_{B(z_0, \frac{r}{2})}} \right) ; \widehat{\tau}_{B(z_0, \frac{r}{2})} < \widehat{\zeta} \right] \\ &= c^2 c_1 M_1^4 f(y). \quad \square \end{aligned}$$

It is well known that the scale invariant Harnack inequality implies the Hölder continuity of harmonic function (for example, see Section 2.3.2 of [28]).

**Corollary 4.12.** *Every harmonic function for $\widehat{X}^D$ is Hölder continuous.*

## 5. Martin representation in arbitrary bounded domains

In this section we assume that $D$ is an arbitrary bounded domain. With Proposition 2.1, Theorem 3.1 and Proposition 3.4 in hand, we can use Theorem 3 in [21] to define the Martin boundary for $\widehat{X}^D$. In fact, we have a stronger result. We will state here results for $\widehat{X}^D$ and $X^D$ simultaneously. From now on, we fix a point $x_0 \in D$ throughout this paper.

Define

$$M_D(x, y) := \begin{cases} \dfrac{\overline{G}_D(x, y)}{\overline{G}_D(x_0, y)} = \dfrac{G_D(x, y)}{G_D(x_0, y)}, & \text{if } x \in D \text{ and } y \in D \setminus \{x_0\}, \\ 1_{\{x_0\}}(x), & \text{if } y = x_0 \end{cases}$$

and

$$\widehat{M}_D(x, y) := \begin{cases} \dfrac{\overline{G}_D(y, x)}{\overline{G}_D(y, x_0)} = \dfrac{h_D(x) G_D(x, y)}{h_D(x_0) G_D(x_0, y)}, & \text{if } x \in D \text{ and } y \in D \setminus \{x_0\}, \\ 1_{\{x_0\}}(x), & \text{if } y = x_0. \end{cases}$$

By (A4) and (A6), we know that for each $y \in D \setminus \{x_0\}$ and $\varepsilon > 0$, $M_D(\cdot, y)$ ($\widehat{M}_D(\cdot, y)$ respectively) is a harmonic function with respect to $X^D$ ($\widehat{X}^D$ respectively) in $D \setminus B(y, \varepsilon)$ and for every $x \in D \setminus B(y, \varepsilon)$,

$$\begin{aligned} &M_D(x, y) = \mathbf{E}_x \left[ M_D(X^D_{\tau_{D \setminus B(y,\varepsilon)}}, y) \right] \quad \text{and} \\ &\widehat{M}_D(x, y) = \mathbf{E}_x \left[ \widehat{M}_D(\widehat{X}^D_{\widehat{\tau}_{D \setminus B(y,\varepsilon)}}, y); \widehat{\tau}_{D \setminus B(y,\varepsilon)} < \widehat{\zeta} \right]. \end{aligned} \tag{5.1}$$

Using our Riesz decomposition theorems, the Harnack inequality and the Hölder continuity of harmonic functions, one can follow the arguments in [24] (see also Section 2.7 of [1] or [30]) to show that the process $X^D$ ($\widehat{X}^D$ respectively) has a Martin boundary $\partial_M D$ ($\widehat{\partial}_M D$ respectively) satisfying the following properties.

(M1) $D \cup \partial_M D$ and $D \cup \widehat{\partial}_M D$ are compact metric spaces;

(M2) $D$ is open and dense in $D \cup \partial_M D$ and in $D \cup \widehat{\partial}_M D$ and its relative topology coincides with its original topology;

(M3) $M_D(x, \cdot)$ can be extended to $\partial_M$ uniquely in such a way that, $M_D(x, y)$ converges to $M_D(x, w)$ as $y \to w \in \partial_M D$, the function $M_D(x, w)$ is jointly continuous on $D \times \partial_M D$, and $M_D(\cdot, w_1) \neq M_D(\cdot, w_2)$ if $w_1 \neq w_2$;

(M4) $\widehat{M}_D(x, \cdot)$ can be extended to $\widehat{\partial}_M$ uniquely in such a way that, $\widehat{M}_D(x, y)$ converges to $\widehat{M}_D(x, w)$ as $y \to w \in \widehat{\partial}_M D$, the function $\widehat{M}_D(x, w)$ is jointly continuous on $D \times \widehat{\partial}_M D$, and $\widehat{M}_D(\cdot, w_1) \neq \widehat{M}_D(\cdot, w_2)$ if $w_1 \neq w_2$.

By repeating the argument in the proof of Proposition 5.1 in [17], we have the following.

**Proposition 5.1.** *For every $w \in \partial_M D$ ($w \in \widehat{\partial}_M D$ respectively), $x \mapsto M_D(x, w)$ ($x \mapsto \widehat{M}_D(x, w)$ respectively) is harmonic with respect to $X^D$ ($\widehat{X}^D$ respectively).*

**Proof.** We include the proof here for $\widehat{X}^D$ for the reader's convenience. Fix $w \in \partial_M D$ and a relatively compact open sets $U \subset \overline{U} \subset U_1 \subset \overline{U_1}$ in $D$. Let $\delta := \frac{1}{2}\text{dist}(U, \partial U_1)$. Choose a sequence $\{y_n\}_{n\geq 1}$ in $D \setminus \overline{U_1}$ converging to $w$ in $D \cup \partial_M D$ so that

$$\widehat{M}_D(x, w) = \lim_{n\to\infty} \widehat{M}_D(x, y_n).$$

Since $\widehat{M}_D(\cdot, y_n)$ is harmonic in a neighborhood of $U$ for every $n \geq 1$, we have

$$\mathbf{E}_x \left[\widehat{M}_D(\widehat{X}^D_{\tau_U}, y_n)\right] = \widehat{M}_D(x, y_n), \quad x \in U.$$

Using the Harnack inequality (Theorem 4.11), we have for every $z \in \partial U$,

$$\widehat{M}_D(z, y_n) = \frac{\overline{G}_D(y_n, z)}{\overline{G}_D(y_n, x_0)} \leq c_1 \frac{\overline{G}_D(y_n, x_0)}{\overline{G}_D(y_n, x_0)} = c_1, \quad n \geq 1,$$

for some $c_1 = c_1(\delta, D) \in (0, \infty)$. Thus by the bounded convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}_x \left[\widehat{M}_D(\widehat{X}^D_{\widehat{\tau}_U}; y_n); \widehat{\tau}_U < \widehat{\zeta}\right] &= \mathbf{E}_x \left[\widehat{M}_D(\widehat{X}^D_{\widehat{\tau}_U}, w); \widehat{\tau}_U < \widehat{\zeta}\right] \\ &= \widehat{M}_D(x, w), \quad x \in U. \quad \Box\end{aligned}$$

Recall that a positive harmonic function $u$ with respect to $X^D$ ($\widehat{X}^D$ respectively) is said to be minimal if $v$ is positive harmonic with respect to $X^D$ ($\widehat{X}^D$ respectively) and $v \leq u$ imply that $v$ is a constant multiple of $u$. The minimal Martin boundaries of $X^D$ and $\widehat{X}^D$ are defined as

$$\partial_m D = \{z \in \partial_M D : M_D(\cdot, z) \text{ is minimal harmonic with respect to } X^D\}$$

and

$$\widehat{\partial}_m D = \{z \in \widehat{\partial}_M D : \widehat{M}_D(\cdot, z) \text{ is minimal harmonic with respect to } \widehat{X}^D\}$$

respectively. Since $M_D(x_0, y) = 1$ for every $y \in (D \cup \partial_M D) \setminus \{x_0\}$, using the Harnack inequality and the Hölder continuity of harmonic functions, we can show that, for any compact subset $K$ of $D$, the family $\{M_D(\cdot, w) : w \in \partial_M D\}$ and $\{\widehat{M}_D(\cdot, w) : w \in \partial_M D\}$ are uniformly bounded and equicontinuous on $K$. One can then apply the Ascoli–Arzelà theorem to prove that, for every excessive function $f$ of $X^D$, there exist a unique Radon measure $\nu_1$ on $D$ and a unique finite measure $\nu_2$ on $\partial_m D$ such that

$$f(x) = \int_D G_D(x, y)\nu_1(\mathrm{d}y) + \int_{\partial_m D} M_D(x, z)\nu_2(\mathrm{d}z), \tag{5.2}$$

and $f$ is harmonic in $D$ with respect to $X$ if and only if $\nu_1 = 0$. Similarly, for every excessive function $f$ of $\widehat{X}^D$, there exist a unique Radon measure $\mu_1$ on $D$ and a unique finite measure $\mu_2$ on $\widehat{\partial}_m D$ such that

$$f(x) = \int_D \overline{G}_D(y,x)\mu_1(\mathrm{d}y) + \int_{\widehat{\partial}_m D} \widehat{M}_D(x,z)\mu_2(\mathrm{d}z), \tag{5.3}$$

and $f$ is harmonic with respect to $\widehat{X}^D$ if and only if $\mu_1 = 0$ (see Section 2.7 of [1]).

## 6. Martin boundary and Boundary Harnack principle for $\widehat{X}^D$

In this section, we will, under some assumption on the domain, identify the Martin boundary of the dual process with the Euclidean boundary and prove a boundary Harnack principle for the dual process.

Recall that a bounded domain $D$ is said to be Lipschitz if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a Lipschitz function $\phi_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying $|\phi_Q(x) - \phi_Q(z)| \le \Lambda |x - z|$, and an orthonormal coordinate system $CS_Q$ with origin at $Q$ such that

$$B(Q,r_0) \cap D = B(Q,r_0) \cap \{y = (y_1,\ldots,y_{d-1},y_d) =: (\tilde{y}, y_d) \text{ in } CS_Q : y_d > \phi_Q(\tilde{y})\}.$$

The pair $(r_0, \Lambda)$ is called the characteristics of the Lipschitz domain $D$.

We first recall the scale invariant boundary Harnack principle for $X^D$ in bounded Lipschitz domains from [17] which is Theorem 4.6 in [17].

**Theorem 6.1.** *Suppose $D$ is a bounded Lipschitz domain. Then there exist constants $M_2, c > 1$ and $r_2 > 0$, depending on $\mu$ only via the rate at which* $\max_{1\le i\le d}$ *goes to zero such that for every $Q \in \partial D$, $r < r_2$ and any non-negative functions $u$ and $v$ which are harmonic with respect to $X^D$ in $D \cap B(Q, M_2 r)$ and vanish continuously on $\partial D \cap B(Q, M_2 r)$, we have*

$$\frac{u(x)}{v(x)} \le c\,\frac{u(y)}{v(y)} \quad \textit{for any } x, y \in D \cap B(Q,r). \tag{6.1}$$

(See [5,15] for boundary Harnack principles for diffusion with no singular drift term.)

It is well known that for diffusions, the scale invariant boundary Harnack principle can be used to prove the Hölder continuity of the ratio of two harmonic functions vanishing continuously near the boundary. We omit the proof of the next lemma. The proof can be found in [1] (see [4] for the extension to jump processes).

**Lemma 6.2.** *Suppose $D$ is a bounded Lipschitz domain. Then there exist positive constants $r_2$ and $M_2$ depending on $D$ such that for any $Q \in \partial D$, $r < r_2$ and non-negative functions $u, v$ which are harmonic with respect to $X^D$ in $D \cap B(Q, M_2 r)$ and vanish continuously on $\partial D \cap B(Q, M_2 r)$, the limit* $\lim_{D\ni x\to w} u(x)/v(x)$ *exists for every $w \in \partial D \cap B(Q,r)$.*

In this section we consider two bounded domains $U$ and $D$ with $U \subset D$. We will not exclude the case $U = D$. Let $x_U \in U$ (if $U = D$, $x_U = x_0$) and define

$$\widehat{M}_{D,U}(x,y) := \begin{cases} \dfrac{h_D(x_U) G_U(y,x)}{h_D(x) G_U(y,x_U)}, & \text{if } x \in U \text{ and } y \in U \setminus \{x_U\}, \\ 1_{\{x_U\}}(x), & \text{if } y = x_U. \end{cases}$$

Note that $\widehat{M}_{D,U}(x, y) = \widehat{M}_D(x, y)$ if $D = U$. By Theorem 3 in [21], we can define the Martin boundary $\widehat{\partial}_M U$ for the process $\widehat{X}^{D,U}$. Moreover, one can prove that for every $w \in \widehat{\partial}_M D$, $x \mapsto \widehat{M}_{D,U}(x, w)$ is harmonic with respect to $\widehat{X}^D$ in $U$ using an argument similar to that of the proof of Proposition 5.1. Let $\widehat{\partial}_m U$ be the minimal Martin boundary of $\widehat{X}^{D,U}$. We also have the Martin representation: for every harmonic function $f$ of $\widehat{X}^D$ in $U$, there is a unique finite measure $\mu_1$ on $\widehat{\partial}_m U$ such that

$$f(x) = \int_{\widehat{\partial}_m U} \widehat{M}_{D,U}(x, z)\mu_1(\mathrm{d}z). \tag{6.2}$$

Suppose $U$ is a bounded Lipschitz domain. We observe that for $y \neq x_U$,

$$\widehat{M}_{D,U}(x, y) = \frac{h_D(x_U)G_U(y, x)}{h_D(x)G_U(y, x_U)}. \tag{6.3}$$

$G_U(\cdot, x)$ and $G_U(\cdot, x_U)$ are harmonic with respect to $X^U$ near the boundary. Moreover they vanish continuously on the boundary by Theorem 2.6 in [17]. Thus from Lemma 6.2, we immediately get the following.

**Proposition 6.3.** *Suppose $U$ is a bounded Lipschitz domain. Then $\widehat{M}_{D,U}(x, y)$ converges as $y \to w \in \partial U$.*

The proposition above says that the Martin boundary is a subset of $\partial U$. We write the limit above as $\widehat{M}_{D,U}(x, w)$ for $(x, w) \in U \times \partial U$. Let $N_U(x, w) := \lim_{U \ni y \to w} \frac{G_U(y,x)}{G_U(y,x_U)}$ so that

$$\widehat{M}_{D,U}(x, w) = \frac{h_D(x_U)N_U(x, w)}{h_D(x)}.$$

We will show that the (minimal) Martin boundary $\widehat{\partial}_m U$ with respect to $\widehat{X}^{D,U}$ coincides with the Euclidean boundary if $D$ and $U$ are bounded $C^{1,1}$. Recall that $\rho_U(x)$ is the distance between $x$ and $\partial U$.

**Theorem 6.4.** *Suppose $D$ and $U$ are bounded $C^{1,1}$ domains with $U \subset D$. Then there exists a constant $c = c(D, U)$ such that*

$$\frac{1}{c}\frac{\rho_U(x)}{|x - w|^d} \leq \widehat{M}_{D,U}(x, w) \leq c\,\frac{1}{|x - w|^d} \tag{6.4}$$

*and*

$$\frac{1}{c}\frac{1}{|x - w|^d} \leq \widehat{M}_D(x, w) \leq c\frac{1}{|x - w|^d}. \tag{6.5}$$

**Proof.** By the Green function estimates for $X^U$ (Theorem 6.2 in [16]), we have

$$c_1\frac{\rho_U(x)}{|x - w|^d} \leq N_U(x, w) \leq c_1\frac{\rho_U(x)}{|x - w|^d}$$

for some positive constants $c_1$ and $c_2$. Thus by (4.8) and the fact that $\rho_U(x) \leq \rho(x) \leq \mathrm{diam}(D)$

$$\frac{1}{c}\frac{h_D(x_U)\rho_U(x)}{|x - w|^d} \leq \widehat{M}_{D,U}(x, w) \leq c\,\frac{h_D(x_U)}{|x - w|^d}. \quad \square$$

The theorem above implies

**Proposition 6.5.** *If $D$ and $U$ are bounded $C^{1,1}$ domains with $U \subset D$, $\widehat{M}_{D,U}(\cdot, w_1) \neq \widehat{M}_{D,U}(\cdot, w_2)$ if $w_1 \neq w_2$.*

Moreover, one can follow the argument in the proof of Theorem 4.4 [7] and show that $\widehat{M}_{D,U}(x, w)$ is minimal harmonic. Thus the minimal Martin boundary of $\widehat{X}^{D,U}$ is the same as the Euclidean boundary in the case when $D$ and $U$ are bounded $C^{1,1}$ domains with $U \subset D$.

**Theorem 6.6.** *Assume that either $D$ and $U$ are bounded $C^{1,1}$ domains with $U \subset D$ or $U$ is bounded Lipschitz domain with $\overline{U} \subset D$. Then for every harmonic function $f$ of $\widehat{X}^D$ in $U$, there is a unique finite measure $\mu_1$ on $\partial U$ such that*

$$f(x) = \int_{\partial U} \widehat{M}_{D,U}(x, z)\mu_1(\mathrm{d}z), \quad x \in U. \tag{6.6}$$

**Proof.** The case when $D$ and $U$ are bounded $C^{1,1}$ domains with $U \subset D$ has already been dealt with in the paragraph before the theorem. In the case when $U$ is bounded Lipschitz domain with $\overline{U} \subset D$, $\widehat{M}_{D,U}(x, z)$ is comparable to $N_U(x, z)$. One can easily modify the argument in page 193–194 of [1] to prove the theorem. We omit the details. □

Now we are ready to prove the boundary Harnack principle for $\widehat{X}^D$. If $D$ is a bounded $C^{1,1}$ domain, then it is easy to check that there exists $R > 0$ such that for any $x \in \partial D$ and $r \in (0, R)$, $B(x, r) \cap D$ is connected.

**Theorem 6.7** (*Boundary Harnack Principle*). *Suppose $D$ be a bounded $C^{1,1}$ domain in $\mathbf{R}^d$ and $R$ is the constant above. Then for any $r \in (0, R)$ and $z_0 \in \partial D$, there exists a constant $c > 1$ such that for any non-negative harmonic functions $u, v$ in $D \cap B(z_0, r)$ with respect to $\widehat{X}^D$ with $uh_D$ and $vh_D$ vanishing continuously on $\partial D \cap B(z_0, r)$, we have*

$$\frac{u(x)}{v(x)} \le c\frac{u(y)}{v(y)}, \quad \textit{for any } x, y \in D \cap B(z_0, r/2).$$

**Proof.** One can find a bounded $C^{1,1}$ domain $U$ such that $D \cap B(z_0, 2r/3) \subset U \subset \overline{U} \subset D \cap B(z_0, r)$. Fix $x_U \in U$ and let

$$M_2(x, z) := \widehat{M}_{D,U}(x, z) = \lim_{U \ni y \to z} \frac{h_D(x_U)G_U(y, x)}{h_D(x)G_U(y, x_U)}.$$

Since $u, v$ are harmonic in $U$ with respect to $\widehat{X}^D$, by Theorem 6.6, there exist finite measures $\mu_1$ and $\nu_1$ on $\partial U$ such that

$$u(x) = \int_{\partial U} M_2(x, z)\mu_1(\mathrm{d}z) \quad \text{and} \quad v(x) = \int_{\partial U} M_2(x, z)\nu_1(\mathrm{d}z), \quad x \in U.$$

Let $N_2(x, z) := \lim_{U \ni y \to z} \frac{G_U(y,x)}{G_U(y,x_U)}$ so that

$$M_2(x, z) = \frac{h_D(x_U)N_2(x, z)}{h_D(x)}.$$

Let $G^0_U$ be the Green function of the Brownian motion $W$ in $U$. Define the Martin kernel $M_1(x,z)$ for the Brownian motion $W$ in $U$:

$$M_1(x,z) := \lim_{U \ni y \to z} \frac{G^0_U(x,y)}{G^0_U(x_U,y)}.$$

Since $U$ is bounded $C^{1,1}$, by Theorem 7.7 in [16], there exists a constant $c_1 = c_1(x_U, U)$ such that

$$\frac{1}{c_1} M_1(x,z) \le N_2(x,z) \le c_1 M_1(x,z). \tag{6.7}$$

Let

$$u_1(x) := \int_{\partial U} M_1(x,z)\mu_1(\mathrm{d}z) \quad \text{and} \quad v_1(x) := \int_{\partial U} M_1(x,z)\nu_1(\mathrm{d}z), \quad x \in U.$$

By (6.7), we have for every $x \in U$

$$\begin{aligned} \frac{u(x)}{v(x)} &= \frac{\int_{\partial U} M_2(x,z)\mu_1(\mathrm{d}z)}{\int_{\partial U} M_2(x,z)\nu_1(\mathrm{d}z)} = \frac{\int_{\partial U} N_2(x,z)\mu_1(dz)}{\int_{\partial U} N_2(x,z)\nu_1(\mathrm{d}z)} \\ &\le c_1^2 \frac{\int_{\partial U} M_1(x,z)\mu_1(\mathrm{d}z)}{\int_{\partial U} M_1(x,z)\nu_1(\mathrm{d}z)} = c_1^2 \frac{u_1(x)}{v_1(x)} \le c_1^4 \frac{u(x)}{v(x)}. \end{aligned}$$

Since $u_1, v_1$ are harmonic for the Brownian motion $W$ in $U$ and vanish continuously on $\partial U \cap \partial D$, by the boundary Harnack principle for Brownian motion (for example, see [1]),

$$\frac{u_1(x)}{v_1(x)} \le c_2 \frac{u_1(y)}{v_1(y)}, \quad x, y \in D \cap B\left(z_0, \frac{r}{2}\right)$$

for some constant $c_2 > 0$. Thus for every $x, y \in D \cap B(z_0, \frac{r}{2})$

$$\frac{u(x)}{v(x)} \le c_1^2 \frac{u_1(x)}{v_1(x)} \le c_2 c_1^2 \frac{u_1(y)}{v_1(y)} \le c_2 c_1^4 \frac{u(y)}{v(y)}. \quad \square$$

## 7. Schrödinger operator in arbitrary bounded domains

In this section we discuss the Schrödinger operator in arbitrary bounded domains. Using our results in the previous sections we know that the main results in [8] are true for $X^D$ and $\widehat{X}^D$ with the reference measure $\xi_D$. In this section we will use the main results in [8] and state carefully for $X^D$ with respect to the Lebesgue measure.

Recall that a measure $\nu$ on $D$ is said to be a smooth measure of $X^D$ with respect to the reference measure $\xi_D$ if there is a positive continuous additive functional (PCAF in abbreviation) $A$ of $X^D$ such that for all bounded non-negative function $f$ on $D$,

$$\int_D f(x)\nu(\mathrm{d}x) = \lim_{t \downarrow 0} \mathbf{E}_{\xi_D}\left[\frac{1}{t}\int_0^t f(X^D_s)\mathrm{d}A_s\right]. \tag{7.1}$$

The additive functional $A$ is called the positive continuous additive functional of $X^D$ with Revuz measure $\nu$ with the reference measure $\xi_D$. It is known (see [27]) that for any $x \in D$, $\alpha \ge 0$ and bounded non-negative function $f$ on $D$,

$$\mathbf{E}_x \int_0^\infty \mathrm{e}^{-\alpha t} f(X^D_t)\mathrm{d}A_t = \int_0^\infty \mathrm{e}^{-\alpha t}\int_D \overline{q}^D(t,x,y) f(y)\nu(\mathrm{d}y)\mathrm{d}t,$$

and thus we have for any $x \in D, t > 0$ and bounded non-negative function $f$ on $D$,

$$\mathbf{E}_x \int_0^t f(X_s^D)\mathrm{d}A_s = \int_0^t \int_D \overline{q}^D(s,x,y)f(y)\nu(\mathrm{d}y)\mathrm{d}s.$$

Therefore by the monotone convergence theorem we have for any $x \in D, t > 0$ and non-negative function $f$ on $D$,

$$\mathbf{E}_x \int_0^t f(X_s^D)\mathrm{d}A_s = \int_0^t \int_D \overline{q}^D(s,x,y)f(y)\nu(\mathrm{d}y)\mathrm{d}s. \tag{7.2}$$

For a signed measure $\nu$, we use $\nu^+$ and $\nu^-$ to denote its positive and negative parts respectively. If $\nu^+$ and $\nu^-$ are smooth measures of $X^D$ with respect to the reference measure $\xi_D$ and $A^+$ and $A^-$ are PCAFs of $X^D$ with Revuz measures $\nu^+$ and $\nu^-$ respectively with respect to the reference measure $\xi_D$, then we call $A := A^+ - A^-$ of $X^D$ the continuous additive functional of $X^D$ with (signed) Revuz measure $\nu$ with respect to the reference measure $\xi_D$.

A measure $\eta$ on $D$ is said to be a smooth measure of $X^D$ with respect to the Lebesgue measure if $h_D\eta$ is a smooth measure of $X^D$ with respect to the reference measure $\xi_D$. From now on, whenever we speak of a smooth measure of $X^D$, we mean a smooth measure of $X^D$ with respect to the Lebesgue measure unless explicitly mentioned otherwise. If $\eta$ is a smooth measure of $X^D$ and $A$ is the PCAF of $X^D$ with Revuz measure $h_D\eta$ with respect to the reference measure $\xi_D$, then by (7.2), we have any $x \in D, t > 0$ and bounded non-negative function $f$ on $D$,

$$\mathbf{E}_x \int_0^t f(X_s^D)\mathrm{d}A_s = \int_0^t \int_D q^D(s,x,y)f(y)\eta(\mathrm{d}y)\mathrm{d}s. \tag{7.3}$$

The additive functional $A$ in the equation above is called the PCAF of $X^D$ with Revuz measure $\eta$ with respect to the Lebesgue measure. From now on, whenever we speak of an additive functional with a given Revuz measure, we mean an additive functional with a given Revuz measure with respect to the Lebesgue measure unless explicitly mentioned otherwise.

Recall $\mathbf{K}_{d,2}$ from the Definition 1.1.

**Proposition 7.1.** *Any measure $\nu$ in $\mathbf{K}_{d,2}$ is a smooth measure of $X^D$ with respect to the Lebesgue measure, or equivalently, $h_D\nu$ is a smooth measure of $X^D$ with respect to $\xi_D$.*

**Proof.** By the definition of $\mathbf{K}_{d,2}$ we can easily check that the function

$$\overline{G}_D(h_D\nu)(x) = G_D\nu(x)$$

is bounded continuous in $D$. Thus, by Definition IV.3.2 of [3], $\overline{G}_D(h_D\nu)$ is a regular potential. Moreover $X^D$ and $\widehat{X}^D$ are Hunt processes with the strong Feller property and they are in the strong duality with respect to $\xi_D$ (Theorem 3.1 and Proposition 3.4). Consequently $h_D\nu$ charges no semi-polar set by Theorem VI.3.5 of [3]. Now we can apply Theorem VI.1 of [27] to conclude that $h_D\nu$ is a smooth measure of $X^D$ with respect to $\xi_D$, or equivalently, $\nu$ is a smooth measure of $X^D$ with respect to the Lebesgue measure. □

If $\nu$ is a signed measure on $D$ such that $\nu^+$ and $\nu^-$ are smooth measures of $X^D$ and if $A^+$ and $A^-$ are the PCAF of $X^D$ with Revuz measures $\nu^+$ and $\nu^-$ respectively, we call $A := A^+ - A^-$ of $X^D$ the continuous additive functional of $X^D$ with (signed) Revuz measure $\nu$.

We recall the definitions of the class of measures from [6,9] and specify it for $X^D$ with the reference measure $\xi_D$ in an bounded domain $D$. We also give a definition for a class of smooth measures with respect to the Lebesgue measure. In the following, $d$ denotes the diagonal of $D \times D$.

**Definition 7.2.** Let $\nu$ be a signed smooth measure of $X^D$ with respect to $\xi_D$ and define $|\nu| := \nu^+ + \nu^-$. $\nu$ is said to be in the class $\mathbf{S}_\infty^{\xi_D}(X^D)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all $(x, z) \in (D \times D) \setminus d$,

$$\int_{D\setminus K} \frac{\overline{G}_D(x, y)\overline{G}_D(y, z)}{\overline{G}_D(x, z)}|\nu|(\mathrm{d}y) = \int_{D\setminus K} \frac{G_D(x, y)G_D(y, z)}{G_D(x, z)}\frac{|\nu|(\mathrm{d}y)}{h_D(y)} \le \varepsilon \tag{7.4}$$

and for all measurable set $B \subset K$ with $|\nu|(B) < \delta$ all $(x, z) \in (D \times D) \setminus d$,

$$\int_{B} \frac{\overline{G}_D(x, y)\overline{G}_D(y, z)}{\overline{G}_D(x, z)}|\nu|(\mathrm{d}y) = \int_{B} \frac{G_D(x, y)G_D(y, z)}{G_D(x, z)}\frac{|\nu|(\mathrm{d}y)}{h_D(y)} \le \varepsilon. \tag{7.5}$$

A function $q$ is said to be in the class $\mathbf{S}_\infty^{\xi_D}(X^D)$ if $\nu(\mathrm{d}x) := q(x)\xi_D(\mathrm{d}x)$ is in the corresponding space.

A signed smooth $\nu$ of $X^D$ is said to be in the class $\mathbf{S}_\infty(X^D)$ if $h_D(x)\nu(\mathrm{d}x)$ is in the class $\mathbf{S}_\infty^{\xi_D}(X^D)$.

For a continuous additive functional $A$ of $X^D$ with Revuz measure $\nu$, we define $e_A(t) = \exp(A_t)$, for $t \ge 0$. For $y \in D$, let $X^{D,y}$ denote the $h$-conditioned process obtained from $X^D$ with $h(\cdot) = G_D(\cdot, y)$ and let $\mathbf{E}_x^y$ denote the expectation for $X^{D,y}$ starting from $x \in D$. We will use $\tau_D^y$ to denote the lifetime of the process $X^{D,y}$.

In the remainder of this section, we assume that $\nu \in \mathbf{S}_\infty(X^D)$ and $A$ is the CAF of $X^D$ with Revuz measure $\nu$ (with respect to the Lebesgue measure). Note that $A$ is also the CAF of $X^D$ with Revuz measure $h_D(x)\nu(\mathrm{d}x)$ with respect to the reference measure $\xi_D$.

The CAF $A$ gives rise to a Schrödinger semigroup

$$Q_t^D f(x) := \mathbf{E}_x\left[e_A(t)f(X_t^D)\right].$$

The function $x \mapsto \mathbf{E}_x[e_A(\tau_D)]$ is called the gauge function of $\nu$. We say $\nu$ is *gaugeable* if $\mathbf{E}_x[e_A(\tau_D)]$ is finite for some $x \in D$. From now on we will assume that $\nu$ is gaugeable. It follows from [6,9] that the gauge function $x \mapsto \mathbf{E}_x\,[e_A(\tau_D)]$ is bounded on $D$. Note that since $h_D(x)\nu(\mathrm{d}x) \in \mathbf{S}_\infty^{\xi_D}(X^D)$, it follows again from [6,9] that

$$\sup_{(x,y)\in(D\times D)\setminus d} \mathbf{E}_x^y\left[|A|_{\tau_D^y}\right] < \infty$$

(see [6]) and therefore by Jensen's inequality

$$\inf_{(x,y)\in(D\times D)\setminus d} \mathbf{E}_x^y[e_A(\tau_D^y)] > 0. \tag{7.6}$$

By Lemma 3.5 of [6], the Green function for the Schrödinger semigroup $\{Q_t^D, t \ge 0\}$ with respect to $\xi_D$ is

$$\overline{V}_D(x, y) = \mathbf{E}_x^y\left[e_A(\tau_D^y)\right]\overline{G}_D(x, y), \tag{7.7}$$

that is,

$$\int_D \overline{V}_D(x, y)f(y)\xi_D(\mathrm{d}y) = \int_0^\infty Q_t^D f(x)\mathrm{d}t = \mathbf{E}_x\left[\int_0^\infty e_A(t)f(X_t^D)\mathrm{d}t\right] \tag{7.8}$$

for any Borel function $f \geq 0$ on $D$. Let

$$V_D(x, y) := \overline{V}_D(x, y) h_D(y)$$

so that

$$\int_D V_D(x, y) f(y) \mathrm{d}y = \int_0^\infty Q_t^D f(x) \mathrm{d}t = \mathbf{E}_x \left[ \int_0^\infty e_A(t) f(X_t^D) \mathrm{d}t \right]. \tag{7.9}$$

Thus $V_D(x, y)$ is the Green function for the Schrödinger semigroup $\{Q_t^D, t \geq 0\}$ with respect to the Lebesgue measure. Since $G_D(x, y) = \overline{G}_D(x, y) h_D(y)$, by Theorem 3.6 in [6] and (7.6), $V_D(x, y)$ is comparable to $G_D(x, y)$ on $D \times D \setminus d$, where $d$ denotes the diagonal of $D \times D$. From Theorem 3.4 [8] and the continuity of $h_D$, we see that $V_D(x, y)$ is continuous on $(D \times D) \setminus d$.

Let $u(x, y) := \mathbf{E}_x^y \left[ e_A(\tau_D^y) \right]$ for $y \in D$, and define $u(x, w) := \mathbf{E}_x^w \left[ e_A(\tau_D^w) \right]$ for $w \in \partial D$, where $\mathbf{E}_x^w$ is the expectation for the conditional process of $X^D$ obtained through $h$-transform with $h(\cdot) = M(\cdot, w)$. Recall that $\partial_m D$ is the minimal Martin boundary of $X^D$.

**Theorem 7.3.** *The following properties hold.*

(1) *For $w \in \partial_m D$ and $x \in D$, $\lim_{D \ni y \to w} u(x, y) = u(x, w)$. The conditional gauge function $u(x, w)$ is jointly continuous on $D \times \partial_m D$.*

(2) *For every $x \in D$ and $w \in \partial_m D$, $K_D(x, w) := \lim_{D \ni y \to w} \frac{V_D(x,y)}{V_D(x_0,y)}$ exists and is finite. Furthermore,*

$$K_D(x, w) = M_D(x, w) \frac{u(x, w)}{u(x_0, w)}, \tag{7.10}$$

*and so $K(x, w)$ is jointly continuous on $D \times \partial_m D$;*

(3) *Assume $D$ is a bounded $C^{1,1}$ domain and let $\rho_D(x)$ be the distance between $x$ and $\partial D$, then there is a constant $c > 1$ such that*

$$c^{-1} \frac{\rho_D(x)}{|x - w|^d} \leq K_D(x, w) \leq c \frac{\rho_D(x)}{|x - w|^d}. \tag{7.11}$$

**Proof.** (1) and (2) are proved in Theorem 3.4 in [8] (also see Section 6 in [8] for the extension). Estimate (7.11) follows directly from (7.10) above and Theorem 7.7 in [16]. □

**Definition 7.4.** A Borel function $u$ defined on $U$ is said to be $\nu$-harmonic for $X^D$ in an open subset $U$ of $D$ if

$$\mathbf{E}_x \left[ e_A(\tau_B) |u(X_{\tau_B}^D)| \right] < \infty \quad \text{and} \quad \mathbf{E}_x \left[ e_A(\tau_B) u(X_{\tau_B}^D) \right] = u(x), \quad x \in B,$$

for every open set $B$ whose closure is a compact subset of $U$. If $U = D$, then $u$ is said to be $\nu$-harmonic for $X^D$.

The following two theorems are proved in [8]. (Lemma 3.6, Theorems 5.11–5.12, Theorem 5.14–5.16 [8]. Also see Section 6 in [8] for an extension.)

**Theorem 7.5.** *For every $z \in \partial_m D$, $x \mapsto K_D(x, z)$ is a minimal $\nu$-harmonic function of $X^D$. That is, if $h$ is a $\nu$-harmonic function of $X^D$ and $0 \leq h(x) \leq K_D(x, z)$, then $h(x) = c K_D(x, z)$ for some constant $c \leq 1$. Moreover, for $z_1 \neq z_2 \in \partial_m D$, $K_D(\cdot, z_1) \not\equiv K_D(\cdot, z_2)$.*

Recall that a non-negative Borel function $f$ defined on $D$ is said to be $\nu$-excessive for $X^D$ if for every $x \in D$ and $t > 0$, $Q_t^D f(x) \le f(x)$ and

$$\lim_{t \downarrow 0} Q_t^D f(x) = f(x).$$

**Theorem 7.6.** *The minimal Martin boundary for the Schrödinger semigroup $\{Q_t^D, t \ge 0\}$ can be identified with the minimal Martin boundary $\partial_m D$ of $X^D$. Furthermore for every $\nu$-excessive function $f$ of $X^D$ that is not identically infinite, there is a unique Radon measure $\mu_1$ on $D$ and a unique finite measure $\mu_2$ on $\partial D$ such that*

$$f(x) = \int_D V_D(x, y)\mu_1(\mathrm{d}y) + \int_{\partial_m D} K_D(x, z)\mu_2(\mathrm{d}z). \tag{7.12}$$

*Function $f$ is $\nu$-harmonic for $X^D$ if and only $\mu_1 = 0$.*

*Conversely, if $\mu_1$ is a Radon measure in $D$ such that $\int_D V_D(x, y)\mu_1(\mathrm{d}y)$ is not identically infinite and $\mu_2$ is finite measure on $\partial D$, then the function $f$ given by (7.12) is a non-negative $\nu$-excessive function of $X^D$ that is not identically infinite.*

Therefore we conclude that the minimal Martin boundary is stable under Feynman–Kac perturbation if $\nu \in \mathbf{S}_\infty(X^D)$ such the gauge function $x \mapsto \mathbf{E}_x\left[e_A(\tau_D)\right]$ is bounded. Furthermore we see that there is a one-to-one correspondence between the space of excessive functions (the space of positive harmonic functions) of $X^D$ that are not identically infinite and the space of $\nu$-excessive functions (the space of positive $\nu$-harmonic functions, respectively) of $X^D$ that are not identically infinite through measures $\mu_1$ and $\mu_2$.

Since the Martin measure $\mu_2$ is finite and $K_D(x, z)$ is jointly continuous on $D \times \partial_m D$, we have the continuity for $\nu$-harmonic functions of $X^D$.

**Theorem 7.7.** *If $u \ge 0$ is $\nu$-harmonic for $X^D$, then $u$ is continuous in $D$.*

In [17], we have shown that for every bounded Lipschitz domain $D$, there is a one-to-one correspondence between the minimal Martin boundary $\partial_m D$ for $X^D$ and the Euclidean boundary $\partial D$ (see Theorem 5.7 in [17]). Thus from Theorem 7.6, we have

**Theorem 7.8.** *For every bounded Lipschitz domain $D$, the (minimal) Martin boundary for the Schrödinger semigroup $\{Q_t^D, t \ge 0\}$ can be identified with the Euclidean boundary. Furthermore, for every $\nu$-excessive function $f$ of $X^D$ that is not identically infinite, $\partial D$ there is a unique Radon measure $\mu_1$ on $D$ and a unique finite measure $\mu_2$ on $\partial D$ such that*

$$f(x) = \int_D V_D(x, y)\mu_1(\mathrm{d}y) + \int_{\partial D} K_D(x, z)\mu_2(\mathrm{d}z). \tag{7.13}$$

*Function $f$ is $\nu$-harmonic for $X^D$ if and only $\mu_1 = 0$.*

**Remark 7.9.** In [18], by using the Green function estimates and our Proposition 7.1, we show that, in fact, $\mathbf{K}_{d,2}$ is contained in $\mathbf{S}_\infty(X^D)$ if $D$ is bounded Lipschitz (see Theorem 2.17 in [18]).

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